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The Klein–Gordon equation with indefinite form

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Abstract. In this manuscript we consider the Klein–Gordon equation and prove asymptotic completeness for modified Möller operators in the case where the associated energy form is not positive on a subspace of infinite dimension. In particular, the subspace orthogonal in the energy form to the eigenvectors of the Hamiltonian of the Klein–Gordon equation has no singular spectrum.

1. Introduction

In this paper the Klein–Gordon equation is studied. The case where the associated energy form is not positive on a subspace of infinite dimension is considered. We show that for scattering theory the subspace of negative energy is not relevant. Asymptotic completeness for generalized Möller operators is proved. In particular, the subspace orthogonal in the energy form to the eigenvectors of the Hamiltonian of the Klein–Gordon equation has no singular spectrum. See theorem 1 below.

We refer to the papers [5, 7, 8] where it is proved in the general setting of Lax–Phillips asymptotic completeness for the acoustic and wave equations for the case where the energy form is positive on a subspace of finite codimension. We refer to [14] where the scattering theory for the Klein–Gordon equation for the case where the energy form is positive is developed. We also refer to [15] where scattering theory for the Klein–Gordon equation is considered for other norms apart from the energy norm. In [14] and [15], the norm induced by the perturbed energy form is equivalent to the norm induced by the non-perturbed energy form. Here we do not assume this.

In section 2 we define the spaces $\mathcal{H}_E, \mathcal{H}_0$ and the Hamiltonians H, H_0 for the Klein–Gordon equation in the perturbed and the free cases. It is showed that both operators generate unitary groups with respect to the corresponding energy forms. In section 3 we define the Möller operators for H and H_0. We prove that they exist and show that their range consist of the subspace orthogonal in the energy form to the eigenvectors for H.

2. Klein–Gordon equation Hamiltonians

The Klein–Gordon equation is the partial differential equation

$$-\frac{\partial^2}{\partial t^2}\Psi(x,t) = \left[\sum_{j=1}^{n}(D_j - b_j)^2 + m^2 + q_s(x)\right]\Psi(x,t) \tag{2.1}$$

with $x \in R^n, n \geqslant 1,\ t \in R^1$ and

$$D_j := -\mathrm{i}\frac{\partial}{\partial x_j} \qquad j = 1, 2, \ldots, n. \tag{2.2}$$

For $A(x)$ a measurable function on R^n, we denote by A the multiplication operator with the function $A(x)$ as well as the function itself. q_s and $b_j,\ j = 1, 2, \ldots, n$ are real-valued measurable functions on R^n and m is a non-negative constant.

This equation describes a relativistic particle with spin zero and mass m in the presence of a magnetic potential $b_j,\ j = 1, 2, \ldots, n$ and scalar potential q_s.

By following a common procedure one can transform equation (2.1) to an equivalent equation of order one with respect to time. We take $f_1(x, t) = \Psi(x, t)$ and $f_2(x, t) = \mathrm{i}\partial\Psi(x, t)/\partial t$ and define

$$\overline{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{2.3}$$

(2.1) is equivalent to the following equation

$$\mathrm{i}\frac{\partial}{\partial t}\overline{f} = h\overline{f} \tag{2.4}$$

where h is the matrix operator given by

$$h = \begin{bmatrix} 0 & 1 \\ L & 0 \end{bmatrix} \tag{2.5}$$

$$L = \sum_{j=1}^{n}(D_j - b_j)^2 + m^2 + q_s(x). \tag{2.6}$$

Equations (2.4)–(2.6) for $q_s = b_j = 0,\ j = 1, 2, \ldots, n$ are respectively

$$\mathrm{i}\frac{\partial}{\partial t}\overline{f} = H_0\overline{f} \tag{2.7}$$

$$H_0 = \begin{bmatrix} 0 & 1 \\ L_0 & 0 \end{bmatrix} \tag{2.8}$$

$$L_0 = -\Delta + m^2. \tag{2.9}$$

Here

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{2.10}$$

$(\ ,\ )$ will always denote the usual inner product on $L^2(R^n, \mathrm{d}^n x) \equiv L^2(R^n)$ antilinear in the factor on the left. We can associate with L_0 a sequilinear form defined for $\overline{f}, \overline{g} \in C_0^{\infty,2}(R^n) := C_0^\infty(R^n) \oplus C_0^\infty(R^n)$

$$\overline{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{2.11}$$

$$\overline{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \tag{2.12}$$

$$(\overline{f}, \overline{g})_0 = \sum_{j=1}^{n}(D_j f_1, D_j g_1) + m^2(f_1, g_1) + (f_2, g_2). \tag{2.13}$$

$C_0^{\infty,2}(R^n)$ is a pre-Hilbert space with inner product $(\, , \,)_0$. Let $\mathcal{H}_0$ denote the completion of $C_0^{\infty,2}(R^n)$ with respect to the norm induced by this inner product. A direct calculation shows that

$$\mathcal{H}_0 \equiv H_1 \oplus L^2(R^n) \tag{2.14}$$

where $H_s := \{\varphi \in L^2(R^n) | \, \|(1+|p|^2)^{s/2} F\varphi\| < \infty\} \equiv$ Sobolev space of order s, if $m \neq 0$. $H_1 := \overline{D(L_0^{1/2})}^0$, if $m = 0$. Here $D(A)$ means the domain of the operator A. For a non-negative self-adjoint operator A, $A^{1/2}$ will denote the positive square root of A defined by functional calculus. $\overline{M}^0$ denotes the completion of M with respect to the norm induced by the inner product $(\, , L_0 \,)$ and F denotes the Fourier transform operator defined as an unitary operator on $L^2(R^n)$ [10]

$$(Ff)(k) := \lim_{M\to+\infty} \frac{1}{(2\pi)^{n/2}} \int_{|x|\leqslant M} e^{-ik\cdot x} f(x) \, d^n x. \tag{2.15}$$

Here $\lim_{M\to+\infty}$ means the limit in the L^2-norm. In general, there are ideal vectors $\overline{f} = \binom{f_1}{f_2} \in \mathcal{H}_0$ with f_1 not belonging to $L^2(R^n)$.

The case $m \neq 0$ is easier to treat. In fact, some of the arguments given below simplify. We only give the details for $m = 0$.

Let $\{P_\Omega(K)\}$ denote the spectral family associated with a self-adjoint operator K. A dense set in $\mathcal{H}_0$ is given by vectors $\overline{f} = \binom{f_1}{f_2}$ such that for some $\delta > 0$

$$f_1 \in D(L_0^{1/2}) \tag{2.16}$$

$$f_2 \in L^2(R^n) \tag{2.17}$$

$$P_{(\delta,+\infty)}(L_0) f_1 = f_1 \qquad P_{(\delta,+\infty)}(L_0) f_2 = f_2. \tag{2.18}$$

We put $\mathcal{L}_2^2(R^n) := L^2(R^n, d^n x) \oplus L^2(R^n, d^n x)$ and let V_0 be the linear operator from $\mathcal{H}_0$ into $\mathcal{L}_2^2(R^n)$ defined by [13]

$$V_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} L_0^{1/2} & 1 \\ L_0^{1/2} & -1 \end{pmatrix}. \tag{2.19}$$

Here $L_0^{1/2}$ is defined by functional calculus. V_0 is well defined at least for vectors $\overline{f}$ obeying (2.16)–(2.18). For these vectors $\overline{f}$

$$\|V_0 \overline{f}\|^2_{\mathcal{L}_2^2} = \|\overline{f}\|^2_{\mathcal{H}_0}. \tag{2.20}$$

This implies that V_0 is unitary from a dense set in $\mathcal{H}_0$ into $\mathcal{L}_2^2(R^n)$. Furthermore, a dense set in $\mathcal{L}_2^2(R^n)$ is given by vectors $\overline{g} = \binom{g_1}{g_2}$ such that for some $\delta > 0$

$$g_1, g_2 \in L^2(R^n) \tag{2.21}$$

$$P_{(\delta,+\infty)}(L_0) g_1 = g_1 \qquad P_{(\delta,+\infty)}(L_0) g_2 = g_2 \tag{2.22}$$

Let $\overline{g} = \binom{g_1}{g_2}$ obey (2.21)–(2.22). It follows that

$$\overline{g}' = \frac{1}{\sqrt{2}} \begin{pmatrix} L_0^{-1/2}(g_1 + g_2) \\ L_0^{-1/2}(g_1 - g_2) \end{pmatrix} \tag{2.23}$$

is a vector in $D(L_0^{1/2}) \oplus L^2(R^n)$ for which (2.16)–(2.18) are valid and

$$V_0\overline{g}' = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \overline{g}. \tag{2.24}$$

Therefore, (2.20)–(2.24) show that V_0 is unitary from a dense set in $\mathcal{H}_0$ onto a dense set in $\mathcal{L}_2^2(R^n)$. V_0 can be extended to an unitary operator from $\mathcal{H}_0$ onto $\mathcal{L}_2^2(R^n)$. We denote this extension also by V_0. A direct calculation shows that for vectors $\overline{g} = \binom{g_1}{g_2}$ satisfying (2.21) and (2.22), the inverse operator V_0^{-1} of V_0 is

$$V_0^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} L_0^{-1/2} & L_0^{-1/2} \\ 1 & -1 \end{pmatrix}. \tag{2.25}$$

For $f_1 \in \overline{D(L_0^{1/2})}^0$ one has that $L_0^{1/2} f_1 \in L^2(R^n)$. If we denote by $L_0^{-1/2} g_1$ the data $\psi \in \overline{D(L_0^{1/2})}^0$ such that $L_0^{1/2}\psi = g_1 \in L^2(R^n)$ then (2.25) is valid on all of $\mathcal{L}_2^2(R^n)$. By direct calculation we obtain

$$H_0 = V_0^{-1} \hat{H}_0 V_0 \tag{2.26}$$

with

$$\hat{H}_0 = \begin{pmatrix} L_0^{1/2} & 0 \\ 0 & -L_0^{1/2} \end{pmatrix}. \tag{2.27}$$

It follows easily that H_0 is a self-adjoint operator on $\mathcal{H}_0$ with domain $D(H_0) = [D(L_0)] \oplus D(L_0^{1/2})$. Here

$$[D(L_0)] = \{f_1 | f_1 \in \overline{D(L_0^{1/2})}^0, L_0^{1/2} f_1 \in D(L_0^{1/2})\} \tag{2.28}$$

In the above we have used that L_0 is a positive operator on $L^2(R^n)$. If the operator L has also a positive self-adjoint extension on $L^2(R^n)$, one can follow step by step the calculations done above. We consider the case when the operator L has a non-positive self-adjoint extension.

Assumption 1. There exists a compact subset of measure zero $\Gamma \subset R^n$, contained in a ball of radius r_0 such that the functions $q_s, D_j \cdot b_j,\ j = 1, \ldots, n$ belong to $L^p_{\mathrm{loc}}(R^n \backslash \Gamma), p \geqslant 2$ and the functions $b_j, j = 1, \ldots, n$ belong to $L^q_{\mathrm{loc}}(R^n \backslash \Gamma), q \geqslant 4$.

We take for a measurable function $f : R^n \to \mathrm{C}$ the operator $f(-\mathrm{i}\nabla)$ in $L^2(R^n)$ [10] acting by the rule

$$(f(-\mathrm{i}\nabla)\varphi)(x) := (F^{-1} f F\varphi)(x) \tag{2.29}$$

with domain of definition $D(f(-\mathrm{i}\nabla))$

$$D(f(-\mathrm{i}\nabla) = \{\varphi \in L^2(R^n) | f(k) \cdot (F\varphi)(k) \in L^2(R^n)\}. \tag{2.30}$$

Here F^{-1} is the inverse of the Fourier transform F. We also put

$$G_\ell(k) = |k|^{2\ell} + 1. \tag{2.31}$$

Let ϕ be a positive function in $C^\infty(R^+)$ with $\|\phi\|_\infty \leqslant 1$, and such that

$$\phi(x) = \begin{cases} 0 & \text{for } 0 < x < 1 \\ 1 & \text{for } 2 < x < +\infty. \end{cases}$$

J will denote the operator which acts by multiplication with the function

$$j_{r_0}(x) := \phi(|x|/r_0) \tag{2.32}$$

where Γ is contained in the ball of radius r_0, $B_{r_0} \subset R^n$.

Assumption 2. The operator L defined on $C_0^\infty(R^n \setminus \Gamma)$ has a self-adjoint extension on $L^2(R^n)$. We will also denote by L any such extension. This extension obeys the Enss conditions [3]

$$(L - \mathrm{i})^{-1} J - J(L_0 - \mathrm{i})^{-1} \equiv \text{compact operator} \tag{2.33a}$$

$$(1 - J)(L - \mathrm{i})^{-1} \equiv \text{compact operator} \tag{2.33b}$$

and

$$\|(LJ - JL_0)G_q^{-1}(-\mathrm{i}\nabla)F_r^c\| \equiv h(r) \in L^1(R^+, \mathrm{d}r) \qquad \text{for some positive integer } q \tag{2.34}$$

$$\|(LJ - JL_0)G_q^{-1}(-\mathrm{i}\nabla)\| < +\infty. \tag{2.35}$$

Here F_r^c is the characteristic function of the complement of the ball of radius r in R^n.

From Weyl's theorem and (2.33) the essential spectrum of L, $\sigma_{\mathrm{ess}}(L)$, is equal to $[0, +\infty)$. Then the spectrum of L below 0 consists of discrete eigenvalues with finite multiplicity. We put $P := P_{(-\infty,0)}(L)$ and P_0 will denote the spectral projection onto the kernel for L. Moreover, $P^\perp := I - P - P_0$. Here I is the identity operator on $L^2(R^n)$. Let $\{\psi_j\}_{j=1}^{+\infty}$ denote the eigenvectors for L corresponding to the strictly negative eigenvalues $\{-\lambda_j^2\}_{j=1}^{+\infty}$ counting multiplicity

$$L\psi_j = -\lambda_j^2 \psi_j \qquad \lambda_j > 0 \qquad \forall j = 1, 2, \ldots, +\infty \tag{2.36}$$

$$(\psi_j, \psi_k) = \delta_{jk} \qquad \forall j, k = 1, 2, \ldots, +\infty. \tag{2.37}$$

The energy form associated with the Klein–Gordon equation (2.1) is taken to be

$$(\bar{f}, \bar{g})_E = (f_1, Lg_1) + (f_2, g_2) \tag{2.38}$$

$$\bar{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{2.39}$$

$$\bar{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \tag{2.40}$$

This energy form is defined for data $\bar{f}, \bar{g} \in D(|L|^{1/2}) \oplus L^2(R^n)$. We denote by $\mathcal{H}_E^-$ the subspace of $D(|L|^{1/2}) \oplus L^2(R^n)$ generated by the finite linear combinations of data $\bar{f} = \binom{f_1}{0}$, with $f_1 \in D(|L|^{1/2})$ and $Pf_1 = f_1$. Let the vectors $\bar{\psi}_j^\uparrow \in \mathcal{H}_E^-$ be given by

$$\bar{\psi}_j^\uparrow := \begin{pmatrix} \psi_j / \lambda_j \\ 0 \end{pmatrix} \qquad \forall j = 1, 2, \ldots, +\infty. \tag{2.41}$$

We obtain from (2.36) and (2.38) that

$$(\bar{\psi}_j^\uparrow, \bar{\psi}_k^\uparrow)_E = -\delta_{jk} \qquad \forall j, k = 1, 2, \ldots, +\infty. \tag{2.42}$$

The set of vectors $\bar{f} = \binom{f_1}{f_2}$ belonging to the E-orthogonal complement to $\mathcal{H}_E^-$ in $D(|L|^{1/2}) \oplus L^2(R^n)$ must obey

$$(\bar{\psi}_j^\dagger, \bar{f})_E = 0 \qquad \forall j \geqslant 1. \tag{2.43}$$

It follows that $\bar{f}$ obeys (2.43) if and only if

$$(\psi_j, f_1) = 0 \qquad \forall j \geqslant 1. \tag{2.44}$$

Therefore, $\bar{f} = \binom{f_1}{f_2}$ is in the E-orthogonal complement to $\mathcal{H}_E^-$ if and only if

$$f_1 \in (\mathrm{Ran}\ P)^\perp \cap D(|L|^{1/2}) \tag{2.45}$$

$$f_2 \in L^2(R^n). \tag{2.46}$$

Let $\mathcal{H}'_E$ denote the set of data $\bar{f} = \binom{f_1}{f_2}$ obeying (2.45) and (2.46) . On this set of data, the energy form is non-negative but not positive definite if L has eigenvalue zero. A way out of this difficulty is to work in the quotient space $\mathcal{H}'_E/Z_0$, where

$$Z_0 = \{\bar{f} = \tbinom{f_1}{f_2} | f_1 \in \mathrm{Ran}\ P_0,\ f_2 = 0\}. \tag{2.47}$$

We define $\mathcal{H}_E^+ := \mathcal{H}'_E/Z_0$. The coset corresponding to a vector $\bar{f} \in \mathcal{H}_E^+$, will also be denoted by $\bar{f} = \binom{f_1}{f_2}$. The energy form (2.38) is induced on $\mathcal{H}_E^+$. We will use the same symbol $(\ ,\ )_E$ to indicate this induced form.

We denote by $\mathcal{H}_E^s$ the direct sum of $\mathcal{H}_E^-$ and $\mathcal{H}_E^+$. The energy form is negative definite on $\mathcal{H}_E^- \times \mathcal{H}_E^-$ and positive definite on $\mathcal{H}_E^+ \times \mathcal{H}_E^+$. $\mathcal{H}_E$ will denote the direct sum of the spaces obtained from the completion of $\mathcal{H}_E^\pm$ with respect to the norms induced by the forms $\pm(\ ,\ )_E$. Then a vector $\bar{f} = \binom{f_1}{f_2}$ is in $\mathcal{H}_E$, if and only if it can be written as

$$\bar{f} = \bar{g} + \bar{h} \tag{2.48}$$

with

$$\bar{g} = \sum_{j=1}^{+\infty} c_j \bar{\psi}_j^\dagger \qquad \sum_{j=1}^{+\infty} |c_j|^2 < +\infty \tag{2.49}$$

and $\bar{h}$ belongs to the completion of $\mathcal{H}_E^+$ with respect to the norm induced by the inner product

$$(\bar{f}, \bar{g})_+ = (f_1, P^\perp L g_1) + (f_2, g_2). \tag{2.50}$$

The space $\mathcal{H}_E$ coincides with the Hilbert space obtained by completion of the pre-Hilbert space $\mathcal{H}_E^s$ with inner product $(\ ,\ )_{\mathcal{H}}$

$$(\bar{f}, \bar{g})_{\mathcal{H}} := (f_1, |L| g_1) + (f_2, g_2). \tag{2.51}$$

For $\bar{f}, \bar{g} \in \mathcal{H}_E^s$ one can see that

$$|(\bar{f}, \bar{g})_E| \leqslant (\bar{f}, \bar{f})_{\mathcal{H}}^{1/2}\ (\bar{g}, \bar{g})_{\mathcal{H}}^{1/2}. \tag{2.52}$$

Therefore, the energy form $(\ ,\ )_E$ can be extended in an unique way to all of $\mathcal{H}_E \times \mathcal{H}_E$. As in the case for $\mathcal{H}_0$, a dense set for $\mathcal{H}_E$ in the $\mathcal{H}$-norm is given by vectors $\bar{f} = \binom{f_1}{f_2}$ such that for some $\epsilon > 0$

$$f_1 \in D(|L|^{1/2}) \tag{2.53}$$

$$f_2 \in L^2(R^n) \tag{2.54}$$

$$P_{(-\infty,-\epsilon)}(L) f_1 + P_{(\epsilon,+\infty)}(L) f_1 = f_1 \tag{2.55}$$

$$P_{(-\infty,-\epsilon)}(L) f_2 + P_{(\epsilon,+\infty)}(L) f_2 = f_2 \tag{2.56}$$

and by the vectors $\bar{f} = \binom{f_1}{f_2}$ such that

$$f_1 = 0 \tag{2.57}$$

$$f_2 \in \operatorname{Ran} P_0. \tag{2.58}$$

To see that these vectors form a dense set one can note that each vector in $\mathcal{H}_E^s$ can be written as the sum

$$\bar{f} = \begin{pmatrix} Pf_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ Pf_2 \end{pmatrix} + \begin{pmatrix} 0 \\ P_0 f_2 \end{pmatrix} + \begin{pmatrix} P^\perp f_1 \\ P^\perp f_2 \end{pmatrix}. \tag{2.59}$$

This equality is thought of as vectors in $\mathcal{H}_E$. For two subspaces A and B of $\mathcal{H}_E$ we will denote by $A \oplus_E B$ the direct sum of the subspaces A and B if and only if they are orthogonal with respect to the energy form and also with respect to the inner product $(\ ,\ )_{\mathcal{H}}$.

Let $(I-P)D(|L|^{1/2})$ denote the set of vectors $f \in D(|L|^{1/2})$ with $(I-P)f = f$. Here I is the identity operator on $L^2(R^n)$. We take $\overline{D(|L|^{1/2})}$ as the completion of the space generated by vectors f_1 such that $\binom{f_1}{0} \in \mathcal{H}_E^- \oplus_E ((I-P)D(|L|^{1/2}) \oplus 0)/Z_0 \subset \mathcal{H}_E^- \oplus_E \mathcal{H}_E^+$ with respect to the norm induced by the inner product $(\ ,|L|\ )$. It easily follows that

$$\mathcal{H}_E = \overline{D(|L|^{1/2})} \oplus L^2(R^n). \tag{2.60}$$

If $\bar{f} = \binom{f_1}{0} \in \mathcal{H}_E^s$, then

$$\| |L|^{1/2} f_1 \| = \|\bar{f}\|_{\mathcal{H}}. \tag{2.61}$$

Therefore, $|L|^{1/2} f_1 \in L^2(R^n)$ and $|L|^{1/2}$ can be extended to all of $\overline{D(|L|^{1/2})}$. Similarly, let ϕ belong to $L^\infty(R^1)$ and $\phi(L)$ be defined by functional calculus. If $\bar{f} = \binom{f_1}{0} \in \mathcal{H}_E^- \oplus_E ((I-P)D(|L|^{1/2}) \oplus 0)/Z_0$, then $\binom{\phi(L)f_1}{0} \in \mathcal{H}_E^s$ and

$$\| |L|^{1/2} \phi(L) f_1 \| \leqslant \|\phi\|_\infty \|\bar{f}\|_{\mathcal{H}}. \tag{2.62}$$

Therefore, one can extend $\phi(L)$ to all of $\overline{D(|L|^{1/2})}$. We will also denote by $|L|^{1/2}$ and $\phi(L)$ the extensions to all of $\overline{D(|L|^{1/2})}$.

The Hamiltonian associated with the Klein–Gordon equation is

$$H := \begin{pmatrix} 0 & 1 \\ L & 0 \end{pmatrix} \tag{2.63}$$

with domain

$$D(H) = [D(|L|)] \oplus D(|L|^{1/2}) \tag{2.64}$$

where

$$[D(|L|)] := \left\{ f_1 | f_1 \in \overline{D(|L|^{1/2})}, |L|^{1/2} f_1 \in D(|L|^{1/2}) \right\}. \tag{2.65}$$

On $[D(|L|)]$, L acts by the rule

$$L f_1 = \mathrm{sgn}(L)|L|^{1/2}|L|^{1/2} f_1 = |L|^{1/2}\ \mathrm{sgn}(L)|L|^{1/2} f_1. \tag{2.66}$$

Here

$$\mathrm{sgn}(x) := \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \tag{2.67}$$

and $\mathrm{sgn}(L)$ is defined by functional calculus. By using the continuity of the energy form with respect to the inner product $(\ .\ )_{\mathcal{H}}$ and equation (2.62) one can write

$$(\bar{f}, \bar{g})_E = (|L|^{1/2} f_1, \mathrm{sgn}(L)|L|^{1/2} g_1) + (f_2, g_2) \qquad \forall \bar{f}, \bar{g} \in \mathcal{H}_E. \tag{2.68}$$

The operator H is in general symmetric in the energy form but not self-adjoint if L has a non-trivial kernel. This is showed by the following lemma. See also [5–8].

Lemma 2.1. Let H be defined by (2.63)–(2.65) and L be defined on $[D(|L|)]$ by (2.66). Then

$$(H\bar{f}, \bar{g})_E = (\bar{f}, H\bar{g})_E \qquad \forall \bar{f}, \bar{g} \in D(H). \tag{2.69}$$

If for some $\bar{h}, \bar{g} \in \mathcal{H}_E$

$$(H\bar{f}, \bar{g})_E = (\bar{f}, \bar{h})_E \qquad \forall \bar{f} \in D(H) \tag{2.70}$$

then $\bar{g} \in D(H)$ and $\bar{h} - H\bar{g} \in \mathrm{Ker}\, H$.

Proof. In the proof we will denote by A_δ the operator $P_{(-\infty,-\delta)}(L) + P_{(\delta,+\infty)}(L), \forall \delta \geqslant 0$. By using (2.63)–(2.66) a direct calculation shows that (2.69) is valid. Let $\bar{g}, \bar{h} \in \mathcal{H}_E$ obey (2.70). We take $\bar{f} = \binom{0}{f_2}$ with $f_2 \in D(|L|^{1/2})$. Then $A_\delta f_2 \in D(|L|^{1/2}), \forall \delta > 0$. From (2.67), (2.68) and (2.70) we get

$$\begin{aligned}(A_\delta f_2, L g_1) - (A_\delta f_2, h_2) &= \left(\begin{pmatrix} f_2 \\ 0 \end{pmatrix}, \begin{pmatrix} A_\delta\, \mathrm{sgn}(L) g_1 - |L|^{-1} A_\delta h_2 \\ 0 \end{pmatrix} \right)_{\mathcal{H}} \\ &= 0. \end{aligned} \tag{2.71}$$

The vectors $\binom{\psi}{0}$ with $\psi \in D(|L|^{1/2})$ are a dense set for $\overline{D(|L|^{1/2})} \oplus 0$ in the $\mathcal{H}$-norm. It follows that for all $\delta > 0$

$$\begin{pmatrix} A_\delta\, \mathrm{sgn}(L) g_1 \\ 0 \end{pmatrix} = \begin{pmatrix} |L|^{-1} A_\delta h_2 \\ 0 \end{pmatrix}. \tag{2.72}$$

This equality is for vectors in $\mathcal{H}_E$. Applying $|L|^{1/2}$ on both sides of equation (2.72), one gets for all $\delta > 0$

$$|L|^{1/2} A_\delta \,\mathrm{sgn}(L) g_1 = |L|^{-1/2} A_\delta h_2. \tag{2.73}$$

This equality is for vectors in $L^2(R^n)$. Equation (2.73) shows that $|L|^{1/2}[P_{(-\infty,-1)}(L) + P_{(1,+\infty)}(L)]\,\mathrm{sgn}(L) g_1 \in D(|L|^{1/2})$. Since $|L|^{1/2}P_{[-1,1]}(L)$ is a bounded operator on $L^2(R^n)$ then

$$|L|^{1/2}\,\mathrm{sgn}(L) g_1 \in D(|L|^{1/2}). \tag{2.74}$$

This shows that

$$g_1 \in [D(|L|)] \tag{2.75}$$

for $[\mathrm{sgn}(L)]^2 \equiv$ Identity operator on $\overline{D(|L|^{1/2})}$. By using (2.62), (2.66) and (2.73) one obtains

$$\|Lg_1 - A_0 h_2\| \leqslant \sqrt{\delta}\| \,|L|^{1/2} g_1\| + \|[P_{[-\delta,0)}(L) + P_{(0,\delta]}(L)]h_2\| \\ \to 0 \qquad \text{as } \delta \to 0^+. \tag{2.76}$$

If one takes now $f_1 \in L^2(R^n)$ and uses that $|L|^{-1} A_\delta f_1 \in [D(|L|)]$, $\forall \delta > 0$, one gets from (2.70)

$$\begin{aligned}(L|L|^{-1}A_\delta f_1, g_2) &- (|L|^{-1}A_\delta f_1, Lh_1)\\ &= (f_1, A_\delta\,\mathrm{sgn}(L) g_2) - (|L|^{-1/2}A_\delta f_1, |L|^{1/2}A_\delta\,\mathrm{sgn}(L)h_1)\\ &= (f_1, A_\delta\,\mathrm{sgn}(L)(g_2 - h_1))\\ &= 0.\end{aligned} \tag{2.77}$$

Here we have used that

$$[P_{(-\infty,-\delta)}(L) + P_{(\delta,+\infty)}(L)]h_1 \in L^2(R^n) \qquad \forall \delta > 0. \tag{2.78}$$

Equation (2.77) shows that for vectors in $L^2(R^n)$

$$[P_{(-\infty,-\delta)}(L) + P_{(\delta,+\infty)}(L)]h_1 = [P_{(-\infty,-\delta)}(L) + P_{(\delta,+\infty)}(L)]g_2 \qquad \forall \delta > 0. \tag{2.79}$$

By using that $h_1 \in \overline{D(|L|^{1/2})}$, equation (2.78) and the fact that $|L|^{1/2}$ is a closed operator on $L^2(R^n)$ one deduces that $[P_{(-\infty,-\delta)}(L)+P_{(\delta,+\infty)}(L)]h_1 \in D(|L|^{1/2})$. This implies that $g_2 \in D(|L|^{1/2})$. A similar calculation as in equation (2.76) shows that as vectors in $\mathcal{H}_E$

$$\begin{pmatrix} g_2 \\ 0 \end{pmatrix} = \begin{pmatrix} h_1 \\ 0 \end{pmatrix} \in \overline{D(|L|^{1/2})} \oplus 0. \tag{2.80}$$

From (2.76) and (2.80) we obtain

$$\bar{h} - H\bar{g} = \begin{pmatrix} 0 \\ P_0 h_2 \end{pmatrix} \in \mathrm{Ker}\, H. \tag{2.81}$$

This proves the lemma. □

Let $\mathcal{H}_+$ be the closure in the $\mathcal{H}$-norm of the subspace

$$\mathcal{H}_+'' := \left\{ \bar{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{H}_E^s \middle| \begin{pmatrix} P^\perp f_1 \\ P^\perp f_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\}. \tag{2.82}$$

It follows that

$$\mathcal{H}_+ = P^\perp \overline{D(|L|^{1/2})} \oplus \mathrm{Ran}\, P^\perp. \tag{2.83}$$

On $\mathcal{H}_+ \times \mathcal{H}_+$, the energy form $(\ ,\ )_E$ and the inner product $(\ ,\ )_\mathcal{H}$ coincide

$$(\bar{f}, \bar{g})_E = (\bar{f}, \bar{g})_\mathcal{H} \qquad \forall \bar{f}, \bar{g} \in \mathcal{H}_+. \tag{2.84}$$

Moreover, H leaves $\mathcal{H}_+$ invariant for $\bar{f} \in D(H) \bigcap \mathcal{H}_+$

$$H \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} P^\perp f_2 \\ P^\perp L f_1 \end{pmatrix} \in \mathcal{H}_+. \tag{2.85}$$

Corollary 2.1. H restricted to $\mathcal{H}_+$ is a self-adjoint operator with respect to the energy form $(\ ,\ )_E$ with domain

$$P^\perp [D(|L|)] \oplus P^\perp D(|L|^{1/2}) \tag{2.86}$$

Proof. This follows easily from lemma 2.1 and (2.82)–(2.85). □

One can see that H has the eigenvalue μ if and only if the self-adjoint operator on $L^2(R^n)L$ has the eigenvalue μ^2. Then for the negative eigenvalues $\{-\lambda_j^2\}_{j=1}^{+\infty}$ of L we must have $\mu \equiv \mu_j = \pm \mathrm{i}\lambda_j$, $j = 1, 2, \ldots$. In fact, the corresponding eigenvectors for H are given by

$$\bar{f}_j^\pm \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_j/\lambda_j \\ \pm \mathrm{i}\psi_j \end{pmatrix} := \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_j/\lambda_j \\ 0 \end{pmatrix} \pm \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 0 \\ \psi_j \end{pmatrix} \qquad j = 1, 2, \ldots + \infty. \tag{2.87}$$

These vectors obey

$$H \bar{f}_j^\pm = \pm \mathrm{i} \lambda_j \bar{f}_j^\pm \qquad j = 1, 2, \ldots, +\infty. \tag{2.88}$$

Obviously H leaves the subspace generated by the vectors $\bar{f}_j^\pm$, $j = 1, 2, \ldots, +\infty$ invariant. This subspace will be denoted by $\mathcal{G}$. Let $\psi_j^\downarrow$, $j = 1, \ldots, +\infty$ be defined as

$$\bar{\psi}_j^\downarrow := \begin{pmatrix} 0 \\ \psi_j \end{pmatrix} \qquad j = 1, 2, \ldots, +\infty. \tag{2.89}$$

The vectors $\bar{\psi}_j^\uparrow$ (equation (2.41)), $\bar{\psi}_j^\downarrow$ and $\bar{f}_j^\pm$ are related by the following equations

$$\bar{f}_j^\pm = \frac{1}{\sqrt{2}} (\bar{\psi}_j^\uparrow \pm \mathrm{i} \bar{\psi}_j^\downarrow) \qquad j = 1, 2, \ldots, +\infty \tag{2.90}$$

and

$$\bar{\psi}_j^{\uparrow} := \frac{1}{\sqrt{2}}(\bar{f}_j^{+} + \bar{f}_j^{-}) \qquad \bar{\psi}_j^{\downarrow} = \frac{1}{\sqrt{2}i}(\bar{f}_j^{+} - \bar{f}_j^{-}) \qquad j = 1, 2, \ldots, +\infty. \tag{2.91}$$

Equations (2.88)–(2.91) imply that

$$\mathrm{e}^{-\mathrm{i}tH}\bar{f}_j^{\pm} = \mathrm{e}^{\pm\lambda_j t}\bar{f}_j^{\pm} \qquad j = 1, 2, \ldots + \infty. \tag{2.92}$$

$$\mathrm{e}^{-\mathrm{i}tH}\bar{\psi}_j^{\uparrow} = \frac{1}{\sqrt{2}}(\mathrm{e}^{\lambda_j t}\bar{f}_j^{+} + \mathrm{e}^{-\lambda_j t}\bar{f}_j^{-}) = \cosh(\lambda_j t)\bar{\psi}_j^{\uparrow} + \mathrm{i}\sinh(\lambda_j t)\bar{\psi}_j^{\downarrow} \tag{2.93}$$

$$\mathrm{e}^{-\mathrm{i}tH}\bar{\psi}_j^{\downarrow} = \frac{1}{\sqrt{2}\mathrm{i}}(\mathrm{e}^{\lambda_j t}f_j^{+} - \mathrm{e}^{-\lambda_j t}f_j^{-}) = -\mathrm{i}\sinh(\lambda_j t)\bar{\psi}_j^{\uparrow} + \cosh(\lambda_j t)\bar{\psi}_j^{\downarrow} \tag{2.94}$$

for $j = 1, 2, \ldots, +\infty$. Therefore

$$(\mathrm{e}^{-\mathrm{i}tH}\bar{\psi}_j^{\uparrow}, \mathrm{e}^{-\mathrm{i}tH}\bar{\psi}_j^{\uparrow})_E = -\cosh^2(\lambda_j t) + \sinh^2(\lambda_j t) = (\bar{\psi}_j^{\uparrow}, \bar{\psi}_j^{\uparrow})_E = -1 \tag{2.95}$$

$$(\mathrm{e}^{-\mathrm{i}tH}\bar{\psi}_j^{\downarrow}, \mathrm{e}^{-\mathrm{i}tH}\bar{\psi}_j^{\downarrow})_E = -\sinh^2(\lambda_j t) + \cosh^2(\lambda_j t) = (\bar{\psi}_j^{\downarrow}, \bar{\psi}_j^{\downarrow})_E = 1 \tag{2.96}$$

for $j = 1, 2, \ldots, +\infty$. Since the vectors $\bar{\psi}_j^{\uparrow\downarrow}$ generate the same subspace as the $\bar{f}_j^{\pm}$, one can extend $\mathrm{e}^{-\mathrm{i}tH}$ by linearity to finite linear combinations of $\bar{\psi}_j^{\uparrow\downarrow}$. The operator $\mathrm{e}^{-\mathrm{i}tH}$ remains unitary with respect to the energy form on $\mathcal{G}$. In general, $\mathrm{e}^{-\mathrm{i}tH}$ is not defined on all of the closure $\overline{\mathcal{G}}$, in the $\mathcal{H}$-norm, of $\mathcal{G}$. For a vector $\bar{f}$ is in $\overline{\mathcal{G}}$ if and only if

$$\bar{f} = \sum_{j\geqslant 1} c_j\bar{\psi}_j^{\uparrow} + \sum_{j\geqslant 1} d_j\bar{\psi}_j^{\downarrow} \tag{2.97}$$

with

$$\sum_{j\geqslant 1}(|c_j|^2 + |d_j|^2) < +\infty \tag{2.98}$$

then given $t \in R^1$ one can extend $\mathrm{e}^{-\mathrm{i}tH}$ to vectors $\bar{f} \in \overline{\mathcal{G}}$ such that

$$\begin{aligned}||\mathrm{e}^{-\mathrm{i}tH}\bar{f}||_{\mathcal{H}}^2 &= \sum_{j\geqslant 1}|c_j\cosh(\lambda_j t) - \mathrm{i}d_j\sinh(\lambda_j t)|^2 \\ &\quad + \sum_{j\geqslant 1}|\mathrm{i}c_j\sinh(\lambda_j t) + d_j\cosh(\lambda_j t)|^2 < +\infty.\end{aligned} \tag{2.99}$$

We note that if L is bounded below as a self-adjoint operator on $L^2(R^n)$ then $\mathrm{e}^{-\mathrm{i}tH}$ is defined on all of $\overline{\mathcal{G}}$. To see how $\mathrm{e}^{-\mathrm{i}tH}$ acts on the E-orthogonal complement to $\mathcal{G}$ one uses that

$$\mathcal{H}_E = \overline{\mathcal{G}} \oplus_E \mathcal{H}_+ \oplus_E \mathrm{Ker}\, H. \tag{2.100}$$

This follows from (2.59). On a vector $\binom{0}{\psi_0} \in \mathrm{Ker}\, H \equiv 0 \oplus \mathrm{Ran}\, P_0$, the operator $\mathrm{e}^{-\mathrm{i}tH}$ acts as

$$\mathrm{e}^{-\mathrm{i}tH}\begin{pmatrix}0\\ \psi_0\end{pmatrix} = \begin{pmatrix}0\\ \psi_0\end{pmatrix} \qquad \forall t \in R^1. \tag{2.101}$$

The subspace $\mathcal{H}_+$ is a Hilbert space with inner product given by the energy form $(\ ,\ )_E$. H restricted to $\mathcal{H}_+$ with domain $D(H)\bigcap\mathcal{H}_+$ is a self-adjoint operator, due to corollary 2.1. Then e^{-itH} forms a one-parameter group of unitary operators on $\mathcal{H}_+$.

Now we can define the E-unitary transformation that makes H unitarily equivalent to a diagonal operator on $\mathcal{L}_2^2(R^n)$. See also [13]. Let V be defined by

$$V := V_- \oplus V_+ \oplus V_0 : \mathcal{H}_E \to L^2(R^n) \oplus L^2(R^n) \tag{2.102}$$

$$V_+ : \mathcal{H}_+ \to \operatorname{Ran} P^\perp \oplus \operatorname{Ran} P^\perp \subset L^2(R^n) \oplus L^2(R^n) \tag{2.103}$$

$$V_- : \overline{\mathcal{G}} \to \operatorname{Ran} P \oplus \operatorname{Ran} P \subset L^2(R^n) \oplus L^2(R^n) \tag{2.104}$$

$$V_0 : \operatorname{Ker} H \to \operatorname{Ran} P_0 \oplus \operatorname{Ran} P_0 \subset L^2(R^n) \oplus L^2(R^n). \tag{2.105}$$

The operator V acts on each of the subspaces $\overline{\mathcal{G}}, \mathcal{H}_+$ and $\operatorname{Ker} H$ as follows:

$$V_0\begin{pmatrix} 0 \\ \psi_0 \end{pmatrix} := \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_0 \\ -\psi_0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix} \in \operatorname{Ker} H \tag{2.106}$$

$$V_+ := \frac{1}{\sqrt{2}}\begin{pmatrix} |L|^{1/2} & \operatorname{sgn}(L) \\ |L|^{1/2} & -\operatorname{sgn}(L) \end{pmatrix} \tag{2.107}$$

$$V_- := \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} |L|^{1/2} & \operatorname{sgn}(L) \\ |L|^{1/2} & -\operatorname{sgn}(L) \end{pmatrix}. \tag{2.108}$$

A direct calculation shows that

$$V_-\bar{\psi}_j^\uparrow = \left(\frac{1-i}{2}\right)\begin{pmatrix} \psi_j \\ \psi_j \end{pmatrix} \qquad j = 1, 2, \ldots, +\infty \tag{2.109}$$

$$V_-\bar{\psi}_j^\downarrow = \left(\frac{1+i}{2}\right)\begin{pmatrix} -\psi_j \\ \psi_j \end{pmatrix} \qquad j = 1, 2, \ldots, +\infty \tag{2.110}$$

this requires V_- to be bijective, for the closure in $\mathcal{L}_2^2(R^n)$ of the subspace generated by the vectors $\{V_-\bar{\psi}_j^{\uparrow\downarrow}\}_{j=1}^{+\infty}$ is $\operatorname{Ran} P \oplus \operatorname{Ran} P \subset \mathcal{L}_2^2(R^n)$. Furthermore

$$\|V_-\bar{\psi}_j^\uparrow\|_{\mathcal{L}_2^2}^2 = \left\| \left(\frac{1-i}{2}\right)\begin{pmatrix} \psi_j \\ \psi_j \end{pmatrix} \right\|_{\mathcal{L}_2^2}^2 = 1 = \|\bar{\psi}_j^\uparrow\|_{\mathcal{H}}^2 \tag{2.111}$$

$$\|V_-\bar{\psi}_j^\downarrow\|_{\mathcal{L}_2^2}^2 = \left\| \left(\frac{1+i}{2}\right)\begin{pmatrix} -\psi_j \\ \psi_j \end{pmatrix} \right\|_{\mathcal{L}_2^2}^2 = 1 = \|\bar{\psi}_j^\downarrow\|_{\mathcal{H}}^2. \tag{2.112}$$

If $\bar{f} = \binom{f_1}{f_2} \in \mathcal{H}_+$, then

$$\left\| \frac{1}{\sqrt{2}}\begin{pmatrix} |L|^{1/2} & \operatorname{sgn}(L) \\ |L|^{1/2} & -\operatorname{sgn}(L) \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\|_{\mathcal{L}_2^2}^2 = \| |L|^{1/2} f_1\|^2 + \|f_2\|^2 = \|\bar{f}\|_{\mathcal{H}}^2. \tag{2.113}$$

A similar argument as the one given for the operator H_0 can be used to show that V_+ is an unitary operator from $\mathcal{H}_+$ onto $\operatorname{Ran} P^\perp \oplus \operatorname{Ran} P^\perp$. We define on the space $\mathcal{L}_2^2(R^n)$ the sequilinear form $E(\bar{f}, \bar{g})$ given by

$$E(\bar{f}, \bar{g}) = \sum_{j\geqslant 1} \left[\overline{(V_-\bar{\psi}_j^\downarrow, \bar{f})}_{\mathcal{L}_2^2} (V_-\bar{\psi}_j^\downarrow, \bar{g})_{\mathcal{L}_2^2} - \overline{(V_-\bar{\psi}_j^\uparrow, \bar{f})}_{\mathcal{L}_2^2} (V_-\bar{\psi}_j^\uparrow, \bar{g})_{\mathcal{L}_2^2} \right] + (\bar{f}, (I-P)\bar{g})_{\mathcal{L}_2^2}. \tag{2.114}$$

Here we mean by $(I-P)\bar{f} \equiv (I-P)\binom{f_1}{f_2}$ the vector $\binom{(I-P)f_1}{(I-P)f_2}$. By using that the energy form $(\ ,\ )_E$ is negative definite on $\mathcal{H}_E^-$, positive on the E-orthogonal complement to $\mathcal{H}_E^-$ and equations (2.41), (2.89) and (2.109)–(2.114) one obtains

$$(\bar{f},\bar{g})_E = E(V\bar{f},V\bar{g}) \qquad \forall \bar{f},\bar{g} \in \mathcal{H}_E. \tag{2.115}$$

We note that $E(\ ,\ )$ is continuous with respect to the inner product on $\mathcal{L}_2^2(R^n)$. $\mathcal{H}_E$ is then E-unitary equivalent by means of the transformation V to a subspace of $\mathcal{L}_2^2(R^n)$ if this last space is thought of with given sequilinear form $E(\ ,\ )$. The range of V, $\operatorname{Ran} V$, is

$$\operatorname{Ran} V = (\operatorname{Ran} P \oplus \operatorname{Ran} P) \oplus_{\mathcal{L}_2^2} (\operatorname{Ran} P^{\perp} \oplus \operatorname{Ran} P^{\perp}) \oplus_{\mathcal{L}_2^2} T_0. \tag{2.116}$$

Here T_0 denotes the subspace of $\operatorname{Ran} P_0 \oplus \operatorname{Ran} P_0$ given by

$$T_0 = \{\bar{f}_0 \in \operatorname{Ran} P_0 \oplus \operatorname{Ran} P_0 | \bar{f}_0 = \binom{f_0}{-f_0},\ \text{for some} f_0 \in \operatorname{Ran} P_0\}. \tag{2.117}$$

By direct calculation one obtains

$$V_+^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} |L|^{-1/2} & |L|^{-1/2} \\ \operatorname{sgn}(L) & -\operatorname{sgn}(L) \end{pmatrix} \tag{2.118}$$

$$V_-^{-1} = \frac{1}{2} \begin{pmatrix} |L|^{-1/2} & |L|^{-1/2} \\ \operatorname{sgn}(L) & -\operatorname{sgn}(L) \end{pmatrix} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \tag{2.119}$$

$$V_0^{-1} \begin{pmatrix} \psi_0 \\ -\psi_0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix}. \tag{2.120}$$

Here we mean by $|L|^{-1/2}f_1$ the data $\psi \in \overline{D(|L|^{1/2})}$ such that $|L|^{1/2}\psi = f_1$. In fact, if $f_1 \in (\operatorname{Ran} P_0)^{\perp}$, then

$$\lim_{\delta\to 0} \begin{pmatrix} |L|^{-1/2}[P_{(-\infty,-\delta)} + P_{(\delta,+\infty)}]f_1 \\ 0 \end{pmatrix}. \tag{2.121}$$

converges in the $\mathcal{H}$-norm to a vector $\binom{\psi}{0} \in \overline{D(|L|^{1/2})} \oplus 0$ and

$$|L|^{1/2}\psi = f_1. \tag{2.122}$$

It follows that V_+^{-1} and V_-^{-1} as given in (2.118)–(2.119) are valid on all of $\operatorname{Ran} P^{\perp} \oplus \operatorname{Ran} P^{\perp}$ and all of $\operatorname{Ran} P \oplus \operatorname{Ran} P$ respectively.

Let $\hat{H}$ be the operator on $\mathcal{L}_2^2(R^n)$ defined by

$$\hat{H} := \begin{pmatrix} |L|^{1/2} & 0 \\ 0 & -|L|^{1/2} \end{pmatrix} P^{\perp} + \begin{pmatrix} \mathrm{i}|L|^{1/2} & 0 \\ 0 & -\mathrm{i}|L|^{1/2} \end{pmatrix} (I - P^{\perp}) \tag{2.123}$$

with domain $D(\hat{H}) := D(|L|^{1/2}) \oplus D(|L|^{1/2})$. By direct calculation, one gets from (2.63), (2.118)–(2.120) and (2.123) that

$$VD(H) = D(\hat{H}) \cap \operatorname{Ran} V. \tag{2.124}$$

Moreover, $\hat{H}$ leaves invariant $\operatorname{Ran} V$ and

$$H = V^{-1}\hat{H}V. \tag{2.125}$$

It follows that

$$\begin{aligned} e^{-itH} &= V^{-1}e^{-it\hat{H}}V \\ &= \begin{pmatrix} \cos t|L|^{1/2} & -i|L|^{-1/2}\sin t|L|^{1/2} \\ -i|L|^{1/2}\sin t|L|^{1/2} & \cos t|L|^{1/2} \end{pmatrix} \cdot D_+ \\ &\quad + \begin{pmatrix} \cosh t|L|^{1/2} & -i|L|^{-1/2}\sinh t|L|^{1/2} \\ i|L|^{1/2}\sinh t|L|^{1/2} & \cosh t|L|^{1/2} \end{pmatrix} \cdot (I - D_+). \end{aligned} \tag{2.126}$$

Here D_+ denotes the projection from $\mathcal{H}_E$ onto $\mathcal{H}_+$. One also has that

$$(H\bar{f},\bar{g})_E = E(\hat{H}V\bar{f}, V\bar{g}) \qquad \forall f \in D(H) \quad \forall \bar{g} \in \mathcal{H}_E. \tag{2.127}$$

3. Möller operators for H and H_0

We have shown that the Klein–Gordon equation in the free case is equivalent to

$$i\frac{\partial}{\partial t}\Phi = \hat{H}_0\Phi \qquad \Phi \in \mathcal{L}_2^2(R^n) \tag{3.1}$$

whereas in the perturbed case, it is equivalent to the equation

$$i\frac{\partial}{\partial t}\Phi = \hat{H}\Phi \qquad \dot{\Phi} \in \mathcal{L}_2^2(R^n). \tag{3.2}$$

Here $\mathcal{L}_2^2(R^n)$ is thought of with given sequilinear form $E(\ ,\ )$. It will be shown that for scattering theory the subspace of negative energy is not relevant.

Let $\mathcal{H}^{\mathrm{pp}}(\hat{H})$ denote the closure in the $\mathcal{L}_2^2(R^n)$-norm of the subspace generated by the eigenvectors for $\hat{H}$. Equation (2.123) shows that

$$\mathcal{H}^{\mathrm{pp}}(\hat{H}) := \mathcal{H}^{\mathrm{pp}}(|L|^{1/2}) \oplus \mathcal{H}^{\mathrm{pp}}(|L|^{1/2}). \tag{3.3}$$

Here $\mathcal{H}^{\mathrm{pp}}(|L|^{1/2})$ denotes the purely point subspace [9] for $|L|^{1/2}$, where $|L|^{1/2}$ is considered as a self-adjoint operator on $L^2(R^n)$.

For an operator D acting on $L^2(R^n)$ we will also denote by D the operator on $\mathcal{L}_2^2(R^n)$ given by

$$\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}. \tag{3.4}$$

Let A, A_0 be self-adjoint operators on Hilbert spaces $\mathcal{H}_2$ and $\mathcal{H}_1$, respectively and let Z be a bounded operator from $\mathcal{H}_1$ into $\mathcal{H}_2$. The generalized Möller operators $\Omega_\pm(A, A_0; Z)$ are defined to be

$$\Omega_\pm(A, A_0; Z) := s - \lim_{t\to\pm\infty} e^{itA}Ze^{-itA_0}P_{\mathrm{ac}}(A_0). \tag{3.5}$$

Here $s-\lim_{t\to\pm\infty}$ means strong convergence in $\mathcal{H}_2$. $P_{ac}(A_0)$ denotes the orthogonal projection onto $\mathcal{H}^{ac}(A_0)$, the subspace of absolute continuity for the operator A_0. If $Z \equiv 1 \equiv$ Identity operator on $\mathcal{H}_0$, we put $\Omega_\pm(A, A_0) = \Omega_\pm(A, A_0; Z)$. We note that $P^\perp$ projects onto the subspace $\operatorname{Ran} P^\perp \oplus \operatorname{Ran} P^\perp$ and that $P_{ac}(\hat{H}_0) \equiv 1$. The sequilinear form $E(\ ,\ )$ is positive definite on $(\operatorname{Ran} P^\perp \oplus \operatorname{Ran} P^\perp) \times (\operatorname{Ran} P^\perp \oplus \operatorname{Ran} P^\perp)$ and it coincides with the usual $\mathcal{L}_2^2(R^n)$-norm. Then we can define the Möller operators $\Omega_\pm(\hat{H}, \hat{H}_0; P^\perp J)$

$$\Omega_\pm(\hat{H}, \hat{H}_0; P^\perp J) := s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}} P^\perp J \mathrm{e}^{-\mathrm{i}t\hat{H}_0}. \tag{3.6}$$

Lemma 3.1. Suppose assumptions 1 and 2 are satisfied. Then the Möller operators $\Omega_\pm(\hat{H}, \hat{H}_0; P^\perp J)$ exist, are partial isometries on $\mathcal{L}_2^2(R^n)$ and asymptotically complete. This is to say

$$\operatorname{Ran} \Omega_\pm(\hat{H}, \hat{H}_0; P^\perp J) = [\mathcal{H}^{pp}(\hat{H})]^\perp. \tag{3.7}$$

Here $A^\perp$ denotes the orthogonal subspace to A with respect to the sequilinear form $E(\ ,\ )$. Furthermore, 0 (m and $-m$ for $m \neq 0$) is the only possible finite limit point of the set of eigenvalues for $\hat{H}$. Any eigenvalue not equal to zero ($\pm m$ for $m \neq 0$) has finite multiplicity. One also has the following equalities

$$\begin{aligned}\Omega_\pm(\hat{H}, \hat{H}_0; P^\perp J) &= \begin{pmatrix} \Omega_\pm(|L|^{1/2}, L_0^{1/2}; P^\perp J) & 0 \\ 0 & \Omega_\mp(|L|^{1/2}, L_0^{1/2}, P^\perp J) \end{pmatrix} \\ &= \begin{pmatrix} \Omega_\pm(L, L_0) & 0 \\ 0 & \Omega_\mp(L, L_0) \end{pmatrix}\end{aligned} \tag{3.8}$$

and

$$\operatorname{Ran} \Omega_\pm(\hat{H}, \hat{H}_0; P^\perp J) = \mathcal{H}^c(L) \oplus \mathcal{H}^c(L). \tag{3.9}$$

Here $\mathcal{H}^c(L)$ denotes the subspace of continuity for the self-adjoint operator L considered as an operator in $L^2(R^n)$.

Proof. From (2.134) and (2.27) one obtains that

$$\mathrm{e}^{\mathrm{i}t\hat{H}} P^\perp J \mathrm{e}^{-\mathrm{i}t\hat{H}_0} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}t|L|^{1/2}} P^\perp J \mathrm{e}^{-\mathrm{i}tL_0^{1/2}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}t|L|^{1/2}} P^\perp J \mathrm{e}^{\mathrm{i}tL_0^{1/2}} \end{pmatrix}. \tag{3.10}$$

Therefore, the Möller operators $\Omega_\pm(\hat{H}, \hat{H}_0; P^\perp J)$ exist in $\mathcal{L}_2^2(R^n)$ if and only if the operators $\Omega_\pm(|L|^{1/2}, L_0^{1/2}; P^\perp J)$ exist in the $L^2(R^n)$-norm. Let K be an arbitrary compact interval in R^1 and φ be a real-valued function on R^1 such that its Fourier transform function, $F\varphi$, belongs to $C_0^\infty(R^1 \setminus 0)$. By direct calculation one can see that

$$\begin{aligned}&P_K(L)[\hat{H} P^\perp \hat{J} - P^\perp \hat{J} \hat{H}_0]\varphi(\hat{H}_0) \\ &\quad = P_K(L) P^\perp (LJ - JL_0) L_0^{-1/2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \varphi(\hat{H}_0).\end{aligned} \tag{3.11}$$

Here

$$\hat{J} := \frac{1}{2}\begin{pmatrix} |L|^{1/2} & 1 \\ |L|^{1/2} & -1 \end{pmatrix} J \begin{pmatrix} L_0^{-1/2} & L_0^{-1/2} \\ 1 & -1 \end{pmatrix}. \tag{3.12}$$

Due to the assumption on the support of $F\varphi$ the operator

$$L_0^{-1/2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\varphi(\hat{H}_0) \tag{3.13}$$

defined by functional calculus is well defined and bounded. By using Cook's criterion, hypothesis (2.34)–(2.35) and equation (3.11) one shows that [2, 12]

$$s-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}} P_K(L) P^{\perp} \hat{J} \mathrm{e}^{-\mathrm{i}t\hat{H}_0}\varphi(\hat{H}_0) \tag{2.14}$$

exist. One deduces from (2.33*a*) that the following operator is compact [4]

$$P_K(L)P^{\perp}(\hat{J}-J)\varphi(\hat{H}_0) = P_K(L)P^{\perp}(|L|^{1/2}J - JL_0^{1/2})L_0^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\varphi(\hat{H}_0). \tag{3.15}$$

It follows that

$$s-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}} P_K(L)P^{\perp} J \mathrm{e}^{-\mathrm{i}t\hat{H}_0}\varphi(\hat{H}_0) = s-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\hat{H}} P_K(L)P^{\perp}\hat{J}\mathrm{e}^{-\mathrm{i}t\hat{H}_0}\varphi(\hat{H}_0). \tag{3.16}$$

By density arguments and equation (3.16) one obtains that the Möller operators (3.6) exist. Therefore, the Möller operators $\Omega_{\pm}(|L|^{1/2}, L_0^{1/2}, P^{\perp}J)$ also exist in $L^2(R^n)$. By using (2.34)–(2.35) and the fact that $1-J$ is multiplication by a function with compact support one gets that the Möller operators $\Omega_{\pm}(L, L_0)$ exist and that

$$\Omega_{\pm}(L, L_0) = \Omega_{\pm}(L, L_0; P^{\perp}J) = \Omega_{\pm}(|L|, L_0; P^{\perp}J). \tag{3.17}$$

Application of the weak invariance principle for the Möller operators [1] to the pair of self-adjoint operators $|L|$ and L_0 gives

$$\Omega_{\pm}(|L|^{1/2}, L_0^{1/2}; P^{\perp}J) = \Omega_{\pm}(|L|, L_0; P^{\perp}J) \equiv \Omega_{\pm}(L, L_0; P^{\perp}J). \tag{3.18}$$

From (3.10), (3.17) and (3.18) it is obtained that the operators $\Omega_{\pm}(\hat{H}, \hat{H}_0; P^{\perp}J)$ are isometries and that (3.8) holds true. Assumption 2 also implies that the operators $\Omega_{\pm}(L, L_0)$ are asymptotically complete [2–4, 12]. Therefore, $\mathrm{Ran}\,\Omega_{\pm}(L, L_0) = [\mathcal{H}^{\mathrm{pp}}(L)]^{\perp} = [\mathcal{H}^{\mathrm{pp}}(|L|^{1/2})]^{\perp} = \mathcal{H}^{\mathrm{c}}(L)$. This and equation (3.3) imply (3.7) and (3.9). The assertion concerning the point spectrum of $\hat{H}$ also follows from the corresponding for L [2, 3, 12] and the definition of the operator $\hat{H}$. This proves the lemma. □

Now we consider what lemma 3.1 implies in the original operators H and H_0. From (2.26), (2.125) and (3.8) one gets that the two-space Möller operators exist

$$\begin{aligned}\Omega_{\pm}(H, H_0; V^{-1}P^{\perp}JV_0) &:= s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}V^{-1}P^{\perp}JV_0\mathrm{e}^{-\mathrm{i}tH_0} \\ &= s - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}V^{-1}P^{\perp}V_0\mathrm{e}^{-\mathrm{i}tH_0} \\ &= \Omega_{\pm}(H, H_0; V^{-1}P^{\perp}V_0).\end{aligned} \tag{3.19}$$

Here $s - \lim_{t\to\pm\infty}$ means strong convergence in the $\mathcal{H}$-norm. Since

$$\|\Omega_{\pm}(\hat{H}, \hat{H}_0; P^{\perp})\bar{f}\|_{\mathcal{L}_2^2} = \|\bar{f}\|_{\mathcal{L}_2^2} \qquad \forall \bar{f} \in \mathcal{L}_2^2(R^n) \tag{3.20}$$

then

$$\|\Omega_{\pm}(H, H_0; V^{-1}P^{\perp}V_0)\bar{f}\|_{\mathcal{H}} = \|\bar{f}\|_{\mathcal{H}_0} \qquad \forall \bar{f} \in \mathcal{H}_0. \tag{3.21}$$

Let $\mathcal{H}^{\mathrm{pp}}(H)$ denote the closure in the $\mathcal{H}$-norm of the subspace generated by the eigenvectors for H. Then

$$\mathcal{H}^{\mathrm{pp}}(H) = V^{-1}[\mathcal{H}^{\mathrm{pp}}(\hat{H}) \cap \operatorname{Ran} V]. \tag{3.22}$$

We can now prove:

Theorem 1. Suppose assumptions 1 and 2 are satisfied. Then the two-space Möller operators $\Omega_{\pm}(H, H_0; V^{-1}P^{\perp}V_0)$ exist, are isometries from $\mathcal{H}_0$ into $\mathcal{H}$ and asymptotically complete: that is to say

$$\operatorname{Ran}\Omega_{\pm}(H, H_0; V^{-1}P^{\perp}V_0) = [\mathcal{H}^{\mathrm{pp}}(H)]^{\perp}. \tag{3.23}$$

Here $A^{\perp}$ denotes the subspace orthogonal to A with respect to the energy form $(\ ,\ )_E$. Furthermore, 0 (m and $-m$ for $m \neq 0$) is the only possible finite limit point of the set of eigenvalues for H. Any eigenvalue not equal to zero ($\pm m$ for $m \neq 0$) has finite multiplicity.

Proof. This follows from lemma 3.1 and (3.19)–(3.22). □

Equation (3.23) implies that H has no singular spectrum [11]. From (3.21) one obtains that $V_0\mathrm{e}^{-\mathrm{i}tH_0}\bar{f}$ will be concentrated asymptotically in $\operatorname{Ran} P^{\perp} \oplus \operatorname{Ran} P^{\perp}$

$$\lim_{t\to\pm\infty} \|P^{\perp}V_0\mathrm{e}^{-\mathrm{i}tH_0}\bar{f}\|_{\mathcal{L}_2^2} = \|\bar{f}\|_{\mathcal{H}_0}. \tag{3.24}$$

Since the energy form induces a norm $\|\ \|_E \equiv \|\ \|_{\mathcal{H}}$ on $[\mathcal{H}^{\mathrm{pp}}(H)\,]^{\perp}$, theorem 1 says that any vector $\bar{\eta}(\equiv \Omega_{\pm}\bar{f}_{\pm})$ E-orthogonal to $\mathcal{H}^{\mathrm{pp}}(H)$ has evolved in the past and will evolve in the future asymptotically as a free state with respect to the energy

$$\|\mathrm{e}^{-\mathrm{i}tH}\bar{\eta} - V^{-1}P^{\perp}V_0\mathrm{e}^{-\mathrm{i}tH_0}\bar{f}_{\pm}\|_E \to 0 \qquad \text{if} \quad t \to \pm\infty. \tag{3.25}$$

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